Quasiparticles and Projection Operators

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(Received 3 August 1964)

A formal procedure for expressing the T matrix in terms of a reduced T matrix is developed. The reduced T matrix results when the original interaction or propagator which appears in the T-matrix integral equation is replaced by a reduced interaction or propagator. This reduction procedure provides a very neat derivation of the projection-operator formalism of Feshbach and the quasiparticle formalism of Weinberg. The two formalisms are compared. The projection-operator formalism appears to offer some advantages over the quasiparticle formalism. The expressions that appear have a more direct physical interpretation. For the bound-state problem, the projection-operator formalism leads to a perturbation expansion for the energy which is a generalization of the Wigner-Brillouin perturbation expansion. For the problem of using an elementary-particle state to represent a bound state of the system, the projection-operator formalism leads to an exact correspondence instead of the approximate one provided by the quasiparticle formalism. From this result we conclude that the bound state and the elementary-particle state are completely equivalent ways to describe a discrete state of a system.

I. INTRODUCTION

THE T matrix is a convenient quantity to use for calculating the properties of a quantum-mechanical system. Along the positive real axis of the energy plane the T matrix is equal to the scattering amplitude. The poles of the T matrix along the real energy axis comprise the discrete spectrum of the system.

The T matrix is determined by an integral equation whose solution can be formally represented by an infinite series. This series is useless at energies which are close to poles of the T matrix that are associated with bound or virtual states of the system for which the perturbation interaction of the expansion plays an important role. It is therefore desirable to have a way of separating out of the interaction that part which is vital for the nearby bound or virtual states. This part must be treated in a nonperturbative way. A perturbation treatment can then be used for the remaining part of the interaction.

We present here a general formalism for separating the dependence of the T matrix on the "essential" and "nonessential" parts of the interaction. This formalism provides a convenient way to derive the projection operator formalism of Feshbach[1] and the quasiparticle formalism of Weinberg.[2,3]

In Sec. II we present a procedure for expressing the T-matrix T in terms of a reduced T-matrix T_1 and a modified propagator Δ_1. T_1 satisfies an integral equation similar to that for T except that the interaction V or the propagator G is replaced by a reduced interaction V_1 or reduced propagator G_1. T_1 carries the nonessential part of the interaction and can be evaluated in terms of a perturbation expansion. The modified propagator Δ_1 carries the essential part of the interaction.

In Sec. III we show that a particular choice for the reduced propagator leads to the projection operator formalism of Feshbach.[1] In Sec. IV we show that a particular choice for the reduced interaction leads to the quasiparticle formalism of Weinberg.[2,3] In Sec. V the two formalisms are compared. The projection operator formalism leads to expressions that have a more direct physical interpretation. It yields an expansion for the energy of a bound system which is a generalization of the Wigner-Brillouin perturbation expansion.

In Sec. VI we apply the T-matrix reduction procedure to the problem of using an elementary particle state to represent a bound state of the system. Consistent use of the projection operator formalism leads to an exact correspondence between the elementary particle state and the bound state. Weinberg's treatment[2] leads to an approximate correspondence. From this result we conclude that the bound state and the elementary-particle state are completely equivalent ways to describe a discrete state of a system.

II. THE REDUCED T MATRIX

Let H be the Hamiltonian of the system. Then the T matrix for processes originating in an eigenstate of an operator H_0 is the solution of the integral equation

$$T=V+VGT=V+TGV, \tag{1}$$

where

$$G=\lim_{\epsilon\to 0}(W-H_0+i\epsilon)^{-1}, \tag{2}$$

$$V=H-H_0, \tag{3}$$

and W is the eigenvalue of H_0. The reduced T-matrix T_1 results from replacing the propagator G by a "reduced propagator" G_1.

$$T_1=V+VG_1T_1=V+T_1G_1V. \tag{4}$$

T may be written in terms of T_1 by substituting

$$V=T_1(1+G_1T_1)^{-1} \tag{5}$$

* Fellow of the Alfred P. Sloan Foundation.
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[2] S. Weinberg, Phys. Rev. **130**, 776 (1963).
[3] S. Weinberg, Phys. Rev. **131**, 440 (1963).

into the formal solution of Eq. (1).

$$\begin{aligned} T&=V(1-GV)^{-1}\\ &=T_1[1-(G-G_1)T_1]^{-1}\\ &=T_1+T_1\Delta_1T_1. \end{aligned} \tag{6}$$

In the above expression, the "modified propagator" Δ_1 is given by

$$\Delta_1=[1-(G-G_1)T_1]^{-1}(G-G_1). \tag{7}$$

We define the "reduced interaction" V_1 to be an operator that produces the same result when substituted for V as does G_1 when substituted for G.

$$T_1=V_1+V_1GT_1=V_1+T_1GV_1. \tag{8}$$

It follows that

$$T_1=V_1(1-GV_1)^{-1}=(1-VG_1)^{-1}V. \tag{9}$$

Solving for V_1 we find

$$\begin{aligned} V_1&=V[1+(G-G_1)V]^{-1}\\ &=V-V\Gamma_1V, \end{aligned} \tag{10}$$

where the "residual propagator" Γ_1 is given by

$$\Gamma_1=[1+(G-G_1)V]^{-1}(G-G_1). \tag{11}$$

It is a simple matter to express G_1 and Δ_1 in terms of Γ_1. The result is

$$G_1=G-(1-\Gamma_1V)^{-1}\Gamma_1 \tag{12}$$

and

$$\Delta_1=[1-\Gamma_1(V+T_1)]^{-1}\Gamma_1. \tag{13}$$

In summary, we find that the T-matrix T can be written in terms of a reduced T-matrix T_1. The reduced T matrix is the result of using either a reduced propagator G_1 or a reduced interaction V_1.

III. THE PROJECTION OPERATOR FORMALISM

Suppose the unperturbed Hamiltonian H_0 has a discrete spectrum of states as well as a continuous spectrum. Then a small perturbation V may produce a great modification of the system in consequence of the localization of the system when it is in any of the discrete unperturbed states. The "essential" part of the interaction is thus the part causing transitions from one discrete state to another. To separate out this part of the interaction we introduce the projection operator Q onto the discrete unperturbed states. Let $P=1-Q$ be the projection operator that projects onto the part of Hilbert space spanned by the continuous eigenstates of H_0. The separation out of the essential part of the interaction is then realized by taking the reduced propagator to be

$$G_1=GP. \tag{14}$$

The perturbation expansion of the reduced T-matrix T_1 then contains only diagrams for which the intermediate states belong to the continuous spectrum of H_0. The modified propagator is then

$$\begin{aligned} \Delta_1&=(1-GQT_1)^{-1}GQ\\ &=(W-H_0-QT_1)^{-1}Q\\ &=(W-QH-QT_1GPV)^{-1}Q. \end{aligned} \tag{15}$$

By inverting the operator $Q[W-(H+T_1GPV)]Q$ the matrix elements of V between the discrete eigenstates of H_0 contribute in a nonperturbative fashion. This is true even if T_1 is treated to lowest order in perturbation theory. Thus the "essential" part of the interaction receives the special treatment it deserves. In particular, it is to be expected that the zeros of $Q(W-H-T_1GPV)Q$ will correspond to poles of the T-matrix T.

IV. THE QUASIPARTICLE FORMALISM

Suppose the unperturbed Hamiltonian H_0 has only a continuous spectrum, but the perturbation V is so strong that the system has bound states or metastable states. Corresponding to each such state there is a pole of the T-matrix T. If W is near one of these poles, the perturbation expansion of $T(W)$ in V is not valid. In the quasiparticle formalism due to Weinberg[2,3] one chooses a residual propagator of the form

$$\Gamma_1=|\psi(W)\rangle\gamma_1(W)\langle\bar{\psi}(W)| \tag{16}$$

and seeks a choice of ψ, $\bar{\psi}$, and γ_1 such that V_1 is sufficiently reduced in strength from V so that a perturbation expansion in V_1 is feasible.

For this choice of Γ_1 the modified propagator becomes

$$\Delta_1=|\psi\rangle(1-\gamma_1\langle\bar{\psi}|V+T_1|\psi\rangle)^{-1}\gamma_1\langle\bar{\psi}|. \tag{17}$$

Use of the integral equation for T_1 and the definitions of V_1 and Γ_1 allows us to transform this expression into

$$\Delta_1=|\psi\rangle N^{-2}\gamma_1\delta_1\langle\bar{\psi}|, \tag{18}$$

where

$$\begin{aligned} N&=1-\gamma_1\langle\bar{\psi}|V|\psi\rangle\\ \delta_1&=(1-N^{-2}\gamma_1\langle\bar{\psi}|T_1GV_1|\psi\rangle)^{-1}. \end{aligned} \tag{19}$$

Weinberg has shown[3] that the necessary and sufficient condition for the convergence of the perturbation expansion of T in powers of V is that all the eigenvalues of GV lie within the unit circle. Thus the problem is to choose Γ_1 so that all the eigenvalues of GV_1 lie within the unit circle when some of the eigenvalues of GV may not do so. Weinberg proved[3] that the only eigenvalues of GV_1 that differ from those of GV are the eigenvalues χ which are solutions of

$$0=1+\gamma_1\langle\bar{\psi}|VG(1-V\chi^{-1}G)^{-1}V\chi^{-1}|\psi\rangle. \tag{20}$$

At this point it proves convenient to introduce explicitly the eigenstates of GV.

$$\begin{aligned} GV|\psi_\nu(W)\rangle&=\eta_\nu(W)|\psi_\nu(W)\rangle\\ G^+V|\bar{\psi}_\nu(W)\rangle&=\eta_\nu(W)^*|\bar{\psi}_\nu(W)\rangle, \end{aligned} \tag{21}$$

where $\bar{\psi}_\nu$ is the time reverse of ψ_ν. For these states

$$(\eta_\nu-\eta_\lambda)\langle\bar{\psi}_\nu|V|\psi_\lambda\rangle=0. \tag{22}$$

Now, following Weinberg, we choose

$$|\psi\rangle=\Sigma_\nu g_\nu(W)|\psi_\nu(W)\rangle$$
$$|\bar{\psi}\rangle=\Sigma_\nu g_\nu(W)^*|\bar{\psi}_\nu(W)\rangle. \tag{23}$$

Then Eq. (20) becomes

$$0=1+\gamma_1\Sigma_\nu g_\nu{}^2\eta_\nu(\chi-\eta_\nu)^{-1}\langle\bar{\psi}_\nu|V|\psi_\nu\rangle, \tag{24}$$

assuming the various η_ν to be distinct. Now the γ_1 and various g_ν must be chosen so that all the solutions χ of this equation have moduli less than unity.

If there is just one eigenvalue η_1 of GV which is outside the unit circle, then this eigenvalue can be replaced by zero in the spectrum of GV_1 by setting $g_\nu=0$ for $\nu\neq1$ and choosing

$$g_1{}^2\gamma_1\langle\bar{\psi}_1|V|\psi_1\rangle=1. \tag{25}$$

Then it can be shown that

$$\delta_1=(1-\eta_1)^{-1}. \tag{26}$$

To replace several eigenvalues $\eta_1, \eta_2, \cdots, \eta_n$ of GV by zeroes in the spectrum of GV_1, one chooses $g_\nu=0$ for $\nu>n$ and

$$g_\nu{}^2\gamma_1\langle\bar{\psi}_\nu|V|\psi_\nu\rangle=\prod_{\substack{\mu=1\\ \mu\neq\nu}}^{n}(1-\eta_\mu/\eta_\nu)^{-1}. \tag{27}$$

For this choice one finds

$$\delta_1=\prod_{\nu=1}^{n}(1-\eta_\nu)^{-1}. \tag{28}$$

Although the formalism is very attractive, for practical applications it is just as difficult to determine the ψ_ν and η_ν as the exact solutions and energies. Thus in practice one must use crude guesses for the ψ_ν. Secondly, what one would really like to establish is not so much the convergence of the Born series but the fact that the first few terms provide a satisfactory representation for the T matrix.

V. COMPARISON OF THE TWO FORMALISMS

To discuss the quasiparticle formalism we assume that the reduction process has been successful in forcing the poles of the T matrix to appear in Δ_1. These poles will occur at those values of W for which, according to Eq. (17),

$$1=\gamma_1\langle\bar{\psi}|V+T_1|\psi\rangle, \tag{29}$$

or for which, according to Eq. (19),

$$1=\gamma_1(1-\gamma_1\langle\bar{\psi}|V|\psi\rangle)^{-2}\langle\bar{\psi}|T_1GV_1|\psi\rangle. \tag{30}$$

Let us expand the right side of Eq. (30) in powers of V_1 and then write everything in terms of V again. It is easy to show that this gives

$$1-\gamma_1\langle\bar{\psi}|VGV|\psi\rangle=J_2+J_3+J_4+\cdots, \tag{31}$$
$$J_2=\gamma_1\langle\bar{\psi}|VGVGV|\psi\rangle-\gamma_1{}^2\langle\bar{\psi}|VGV|\psi\rangle^2,$$
$$J_3=\gamma_1\langle\bar{\psi}|VGVGVGV|\psi\rangle-2\gamma_1{}^2\langle\bar{\psi}|VGVGV|\psi\rangle\langle\bar{\psi}|VGV|\psi\rangle+\gamma_1{}^3\langle\bar{\psi}|VGV|\psi\rangle^3,$$
$$\cdots.$$

In the projection operator formalism the poles of Δ_1 occur at those values of W for which

$$0=Q(W-H-T_1GPV)Q. \tag{32}$$

Thus to determine the poles of T that are near W one must proceed by first of all finding an H_0 that has discrete states as similar as possible to the bound and metastable states of H in the vicinity of the energy W. Q then will be the projection operator constructed from these discrete states, and Δ_1 will have a pole corresponding to each of these bound states. An expansion of T_1 in powers of GPV allows us to transform Eq. (32) into

$$Q(W-H)Q=I_2+I_3+I_4+\cdots,$$
$$I_2=QVGVQ-QVGQVQ,$$
$$I_3=QVGVGVQ-QVGVGQVG-QVGQVGVQ+QVGQVGQVQ,$$
$$\cdots. \tag{33}$$

Equation (31) for the quasiparticle formalism is to be compared with Eq. (33) of the projection operator formalism.[4] In Eq. (31) the fact that we are using a perturbation expansion in terms of a reduced interaction V_1 reveals itself by yielding in each order of V_1 terms J_i which are in fact differences of counterterms. Similarly, in Eq. (33) the fact that we are using a perturbation expansion in terms of a reduced propagator G_1 manifests itself by yielding in each order of G_1 terms I_i which are in fact differences of counterterms.

Suppose there is only a single pole nearby to be dealt with. In the quasiparticle formalism we take ψ to be the eigenstate of GV whose eigenvalue η is outside the unit circle. At any rate, we take ψ to be the best approximation to this eigenstate we can find. Then we take γ_1 to be $\langle\bar{\psi}|V|\psi\rangle^{-1}$. We see that the left-hand side of Eq. (31) becomes

$$1-\langle\bar{\psi}|V|\psi\rangle^{-1}\langle\bar{\psi}|VGV|\psi\rangle\approx1-\eta, \tag{34}$$

while the right-hand terms are

$$J_2\approx0$$
$$J_3\approx0$$
$$\cdots. \tag{35}$$

[4] Note that even though we started with the same Hamiltonian $H=H_0+V$, the H_0's used in the two formalisms have been chosen differently. In the quasiparticle formalism H_0 was assumed to have a continuous spectrum. In the projection operator formalism the spectrum of H_0 was assumed to include some discrete states.

The projection operator formalism handles this situation by using an eigenstate ψ of H_0 (eigenvalue W_0) which is an approximation to the true eigenstate of H at the pole. Then we take Q to be $|\psi\rangle\langle\psi|$ and Eq. (33) becomes

$$(W-W_0)-\langle\psi|V|\psi\rangle=I_2+I_3+\cdots,$$
$$I_2=\langle\psi|VGPV|\psi\rangle$$
$$=\langle\psi|VGV|\psi\rangle-(W-W_0)^{-1}\langle\psi|V|\psi\rangle^2,$$
$$I_3=\langle\psi|VGPVGPV|\psi\rangle$$
$$=\langle\psi|VGVGV|\psi\rangle+(W-W_0)^{-2}\langle\psi|V|\psi\rangle^3$$
$$-2(W-W_0)^{-1}\langle\psi|V|\psi\rangle\langle\psi|VGV|\psi\rangle. \quad (36)$$

This result will be recognized as the Wigner-Brillouin bound state perturbation expansion. Thus Eq. (33) must represent a generalization of Wigner-Brillouin perturbation theory. If we neglect the terms on the right-hand side of Eq. (33), we are following the procedure used in the theory of the shell model. That is, we are carrying out the diagonalization of the Hamiltonian in a restricted portion of Hilbert space. Thus I_2, I_3, $\cdots$ provide a perturbation expansion of the error that occurs in this procedure.

Comparing Eq. (36) with Eq. (31) we note that to evaluate the lowest order term given by the quasiparticle formalism we must perform an integral second order in V which appears only in the second order correction term of the projection operator formalism. This would appear to indicate that the projection operator formalism can give a more accurate result for the same amount of work.

Let us now compare the expressions for the T matrices that are given by the two formalisms. If we work to the lowest order in V_1 in the quasiparticle formalism, we find

$$T_1\approx V_1=V-V|\psi\rangle\gamma_1\langle\bar{\psi}|V \quad (37)$$

and

$$T\approx V-V|\psi\rangle\gamma_1(\gamma_1-\langle\bar{\psi}|VGV|\psi\rangle^{-1})^{-1}\gamma_1\langle\bar{\psi}|V. \quad (38)$$

If we work to lowest order in GP in the projection operator formalism, we find

$$T_1\approx V+VGPV \quad (39)$$

and

$$T\approx V+VQ(W-QH-QVGPV)^{-1}QV. \quad (40)$$

Near an isolated pole this would be

$$T\approx V+V|\psi\rangle(W-\langle\psi|H+VGPV|\psi\rangle)^{-1}\langle\psi|V. \quad (41)$$

The quantities in the denominator can be immediately identified with familiar concepts. $W_0=\langle\psi|H_0|\psi\rangle$ is the unperturbed resonance energy,

$$\langle\psi|V|\psi\rangle+\mathrm{Re}\langle\psi|V(W_0-H_0+i\epsilon)^{-1}PV|\psi\rangle$$

is the level shift, and $2\,\mathrm{Im}\langle\psi|V(W_0-H_0+i\epsilon)^{-1}PV|\psi\rangle$ is the width.

IV. THE QUASIPARTICLES

The quasiparticle formalism takes its name from the fact that it can be used to introduce fictitious "elementary particle states" to represent actual bound states of the system. The procedure outlined by Weinberg[2] does not produce an exact equivalence between the original system and the system augmented by quasiparticle states. The equivalence exists only in the limit of infinite bare quasiparticle energy.

In this section we show first of all how the T-matrix reduction procedure allows us to add discrete states to the Hilbert space of a system and then modify the Hamiltonian so that no physical predictions are changed. Secondly, we show how the projection operator formalism may be used to force these quasiparticle states to represent actual bound states of the system. The equivalences established will be exact.

Suppose we augment the Hilbert space of a system by adding a discrete state $|\alpha\rangle$, the quasiparticle state. Let

$$\mathbf{Q}=|\alpha\rangle\langle\alpha|=1-\mathbf{P}. \quad (42)$$

Corresponding to the original operators H, H_0, and V we have in the augmented Hilbert space the operators $\mathbf{H}$, $\mathbf{H}_0$, and $\mathbf{V}$ defined so that

$$\mathbf{PHP}=H\mathbf{P}$$
$$\mathbf{PH}_0\mathbf{P}=H_0\mathbf{P}$$
$$\mathbf{PVP}=V\mathbf{P}. \quad (43)$$

We define

$$\mathbf{G}=\lim_{\epsilon\to 0}(W-\mathbf{H}_0+i\epsilon)^{-1}. \quad (44)$$

Now if

$$\mathbf{T}=\mathbf{V}+\mathbf{VGT}, \quad (45)$$

then it will not in general be true that $\mathbf{PTP}=T\mathbf{P}$, where $T=V+VGT$. So we seek a new Hamiltonian

$$\mathbf{H}_1=\mathbf{H}_0+\mathbf{V}_1 \quad (46)$$

for the augmented system such that

$$\mathbf{PT}_1\mathbf{P}=T\mathbf{P}=V\mathbf{P}+VGT\mathbf{P}, \quad (47)$$

where

$$\mathbf{T}_1=\mathbf{V}_1+\mathbf{V}_1\mathbf{G}\mathbf{T}_1. \quad (48)$$

Let $\mathbf{G}_1$ be the reduced propagator associated with $\mathbf{V}_1$. Then

$$\mathbf{T}_1=\mathbf{V}+\mathbf{V}\mathbf{G}_1\mathbf{T}_1, \quad (49)$$

and

$$\mathbf{PT}_1\mathbf{P}=V\mathbf{P}+\mathbf{PV}\mathbf{G}_1\mathbf{T}_1\mathbf{P}. \quad (50)$$

Clearly, the choice for $\mathbf{G}_1$ required to make Eq. (47) valid is

$$\mathbf{G}_1=G\mathbf{P}. \quad (51)$$

This choice corresponds to the projection operator formalism if

$$\mathbf{GP}=G\mathbf{P}. \quad (52)$$

This last property can be established by completing the definition of $\mathbf{H}_0$ in the following way:

$$\mathbf{H}_0 = H_0\mathbf{P} + E_\alpha\mathbf{Q}. \tag{53}$$

The c number E_α will be called the bare quasiparticle energy. The reduced interaction $\mathbf{V}_1$ can now be displayed.

$$\begin{aligned}\mathbf{V}_1 &= \mathbf{V} - \mathbf{V}\boldsymbol{\Gamma}_1\mathbf{V}\\ \boldsymbol{\Gamma}_1 &= (1+\mathbf{GQV})^{-1}\mathbf{GQ}\\ &= (\mathbf{G}^{-1}+\mathbf{QV})^{-1}\mathbf{Q}\\ &= (W - H_0\mathbf{P} - E_\alpha + \mathbf{QV})^{-1}\mathbf{Q}\\ &= |\alpha\rangle(W - E_\alpha + \langle\alpha|\mathbf{V}|\alpha\rangle)^{-1}\langle\alpha|.\end{aligned} \tag{54}$$

Thus we have shown that if we augment the Hilbert space of our system by adding the discrete quasiparticle state $|\alpha\rangle$ and if we replace the Hamiltonian $\mathbf{H} = \mathbf{H}_0 + \mathbf{V}$ by $\mathbf{H}_1 = \mathbf{H}_0 + \mathbf{V}_1$, then the physical properties of the system are unchanged. This result has been established without restricting the value of the quasiparticle bare energy and without specifying the matrix elements of $\mathbf{V}$ that involve the quasiparticle state.

Now we address ourselves to our second objective—to show how the quasiparticle state can be used to represent the properties of one of the bound states of the system. Suppose the system has a bound state associated with the discrete eigenstate $|a\rangle$ of H_0.

$$(E_a - H_0)|a\rangle = 0. \tag{55}$$

We can reduce the T matrix using this state vector. Let

$$Q = |a\rangle\langle a| = 1 - P. \tag{56}$$

Then

$$\begin{aligned}T &= V + VGT\\ T &= V + VGPT_1\\ T &= T_1 + T_1\Delta_1 T_1\\ \Delta_1 &= (1 - GQT_1)^{-1}GQ\\ &= |a\rangle(W - E_a - \langle a|T_1|a\rangle)^{-1}\langle a|.\end{aligned} \tag{57}$$

Here we see very explicitly how the pole of T associated with the discrete state $|a\rangle$ is carried by the modified propagator Δ_1.

We turn now to the augmented system and make a *second* reduction of the T matrix using the projection operator $\mathbf{Q} = |\alpha\rangle\langle\alpha|$. Reducing $\mathbf{T}_1$ gives

$$\begin{aligned}\mathbf{T}_1 &= \mathbf{T}_{11} + \mathbf{T}_{11}\boldsymbol{\Delta}_{11}\mathbf{T}_{11}\\ \mathbf{T}_{11} &= \mathbf{V}_1 + \mathbf{V}_1\mathbf{G}_1\mathbf{T}_{11}\\ \boldsymbol{\Delta}_{11} &= (1 - \mathbf{GQT}_{11})^{-1}\mathbf{GQ}\\ &= (W - H_0\mathbf{P} - E_\alpha\mathbf{Q} - \mathbf{QT}_{11})^{-1}\mathbf{Q}\\ &= |\alpha\rangle(W - E_\alpha - \langle\alpha|\mathbf{T}_{11}|\alpha\rangle)^{-1}\langle\alpha|.\end{aligned} \tag{58}$$

We know that $\mathbf{T}_1$ is equivalent to T in the sense of Eq. (47). In Eq. (57) we see the pole of T due to $|a\rangle$ appears explicitly in Δ_1. In Eq. (58) we see a similar expression associated with $|\alpha\rangle$. Can we use the freedom that remains to us in the definition of E_α and $\mathbf{V}$ to make this correspondence exact?

The first step is to make $\mathbf{T}_{11}$ equivalent to T_1. We seek to arrange matters so that

$$\begin{aligned}\mathbf{PT}_{11}\mathbf{P} &= T_1\mathbf{P}\\ &= \mathbf{PV}_1\mathbf{P} + \mathbf{PV}_1GT_1\mathbf{P}\\ &= V_1\mathbf{P} + V_1GT_1\mathbf{P}.\end{aligned} \tag{59}$$

By inspection we see that what is required is

$$\begin{aligned}\mathbf{PV}_1\mathbf{P} &= V_1\mathbf{P}\\ &= \mathbf{PVP} - \mathbf{PV}\boldsymbol{\Gamma}_1\mathbf{VP}\\ &= V\mathbf{P} - \mathbf{PV}|\alpha\rangle(W - E_\alpha + \langle\alpha|\mathbf{V}|\alpha\rangle)^{-1}\langle\alpha|\mathbf{VP}\\ &= V\mathbf{P} - V|a\rangle(W - E_a + \langle a|V|a\rangle)^{-1}\langle a|V.\end{aligned} \tag{60}$$

This will follow if we require

$$E_a = E_\alpha, \tag{61}$$

and if we complete the definition of $\mathbf{V}$ by requiring

$$\begin{aligned}\mathbf{QVQ} &= \langle a|V|a\rangle\mathbf{Q},\\ \mathbf{QVP} &= |\alpha\rangle\langle a|V,\\ \mathbf{PVQ} &= V|a\rangle\langle\alpha|.\end{aligned} \tag{62}$$

Using Eqs. (59), (58), and (57) we find that equivalence of the pole terms of T and $\mathbf{T}_1$ requires

$$\begin{aligned}&\mathbf{PT}_{11}|\alpha\rangle(W - E_\alpha - \langle\alpha|\mathbf{T}_{11}|\alpha\rangle)^{-1}\langle\alpha|\mathbf{T}_{11}\mathbf{P}\\ &\quad = T_1|a\rangle(W - E_a - \langle a|T_1|a\rangle)^{-1}\langle a|T_1.\end{aligned} \tag{63}$$

From Eq. (62) we can deduce that

$$\langle\alpha|\mathbf{T}_{11}|\alpha\rangle = \langle a|T_1|a\rangle, \tag{64}$$

and

$$\begin{aligned}\mathbf{PT}_{11}|\alpha\rangle &= T_1|a\rangle\\ \langle\alpha|\mathbf{T}_{11}\mathbf{P} &= \langle a|T_1.\end{aligned} \tag{65}$$

We see that the reduced T-matrix $\mathbf{T}_{11}$ in the augmented Hilbert space is now completely equivalent to the reduced T-matrix T_1. Also the modified propagator $\boldsymbol{\Delta}_{11}$ has a pole at the bound state energy. In this way the quasiparticle state can be used to replace in effect a bound state of the system. Note that our final result differs from that of Weinberg in that the bare quasiparticle energy must be set equal to the eigenvalue E_a of H_0 instead of infinity.

In Weinberg's treatment, the reduction of $\mathbf{T}$ to $\mathbf{T}_1$ is carried out by the residual propagator

$$\boldsymbol{\Gamma}_1 = |\alpha\rangle(-E_\alpha + \langle a|V|a\rangle)^{-1}\langle\alpha|,$$

while the reduction of T to T_1 is effected by

$$\Gamma_1 = |a\rangle(-E_\alpha + \langle a|V|a\rangle)^{-1}\langle\alpha|.$$

The reduction of $\mathbf{T}_1$ to $\mathbf{T}_{11}$, however, is effected by the residual propagator

$$\mathbf{\Gamma}_1 = |\alpha\rangle((W-E_\alpha+\langle\alpha|\mathbf{V}|\alpha\rangle)^{-1}\langle\alpha| .$$

These choices correspond to using the quasiparticle formalism for the first two reductions and using the projection operator formalism for the third reduction. As a result one finds that besides Eq. (62) the physical equivalence of the augmented system to the original system requires $W-E_\alpha-\langle\alpha|\mathbf{T}_{11}|\alpha\rangle=-E_\alpha-\langle a|T_1|a\rangle$. Thus the equivalence obtains only in the limit $E_\alpha \gg W$. But it is seen that if the projection operator formalism is used for all three reductions, then the equivalence can be made exact.

We have shown how an elementary particle state can be used to represent a bound state of the system. There is no reason why we cannot reverse the procedure and use a bound state to replace an elementary-particle state. One merely regards Eq. (43) as a complete definition of H, H_0, and V rather than a partial definition of $\mathbf{H}$, $\mathbf{H}_0$, and $\mathbf{V}$. Since $\mathbf{H}_1=\mathbf{H}_0+\mathbf{V}_1$ is now the starting point, $\mathbf{V}$ must be defined in terms of $\mathbf{V}_1$

$$\begin{aligned}\mathbf{V}&=\mathbf{V}_1-\mathbf{V}_1\mathbf{\Gamma}\mathbf{V}_1\\ \mathbf{\Gamma}&=-(1-\mathbf{G}\mathbf{Q}\mathbf{V}_1)^{-1}\mathbf{G}\mathbf{Q}\\ &=|\alpha\rangle(E_\alpha+\langle\alpha|\mathbf{V}_1|\alpha\rangle-W)^{-1}\langle\alpha| .\end{aligned} \tag{66}$$

Equation (62) now plays the role of the definition of the matrix elements of V with respect to the new bound state $|a\rangle$.

$$\begin{aligned}\langle a|V|a\rangle&=\langle\alpha|\mathbf{V}|\alpha\rangle\\ V|a\rangle&=\mathbf{P}\mathbf{V}|\alpha\rangle .\end{aligned} \tag{67}$$

We conclude that it is always possible to interpret the discrete states of a system either as bound states or as elementary-particle states since the formalism presented in this section allows us to switch from the one description to the other.

ACKNOWLEDGMENTS

The author is grateful to M. A. Nagarajan and R. M. Thaler for helpful discussions of the contents of this paper.

PHYSICAL REVIEW VOLUME 136, NUMBER 6B 21 DECEMBER 1964

Asymptotic Behavior of the Scattering Amplitude and Normal and Abnormal Solutions of the Bethe-Salpeter Equation. II*

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(Received 7 August 1964)

The high-energy asymptotic expansion of the Green's function for the scattering of two scalar particles in the crossed channel (t channel) is investigated in the ladder approximation by using the scalar-photon-exchange model with scalar coupling. It is shown that each term of the asymptotic expansion in the t channel exactly corresponds to the solutions of the Bethe-Salpeter equation for bound states if one considers the expansion in powers of $(-t-m^2+v_0)$ instead of $(-t)$, where m and $v_0^{1/2}$ are the internal mass and the constant external mass ($v_0 \neq m^2$), respectively. It is proved that all normal solutions of the Bethe-Salpeter equation appear in this expansion. The problem of whether or not abnormal solutions also appear in this expansion is analyzed in detail. Exact solutions in some special cases are presented and discussed.

1. INTRODUCTION

IN a previous paper,[1] which we shall refer to as I, we obtained the exact solution to the Bethe-Salpeter-type integral equation for the off-the-mass-shell scattering amplitude in the case $\mu'=0$, $\mu=2m$, and $s=0$, where μ' and μ are the exchanged-meson masses in the kernel and in the inhomogeneous term, respectively, s being the invariant energy; and we investigated its high-energy asymptotic expansion in the crossed channel (t channel). It was found there that in our model the leading term and the second one exactly correspond to the normal solutions of the Bethe-Salpeter equation for bound states with $n=l+1$ and $n=l+2$, respectively, but the third term does not correspond to those with $n=l+3$, where n and l are the principal and the azimuthal quantrum numbers, respectively. Since we artificially introduced a massive meson only in the inhomogeneous term in order to avoid infrared divergence, it was not clear whether or not the above result was owing to the introduction of the special meson.

The purpose of the present paper is to discuss the high-energy asymptotic expansion (in the crossed channel) of the Green's function for the scattering of two scalar particles in the Bethe-Salpeter formalism in the case $\mu'=\mu=0$ and s arbitrary. In spite of its similarity to the equation discussed in I, it is, unfortunately, extremely difficult to find the exact solution (in closed

* This work was performed under the auspices of the U. S. Atomic Energy Commission.
[1] N. Nakanishi, Phys. Rev. 135, B1430 (1964).